

# A twisted look on kappa-Minkowski: $U(1)$ gauge theory

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## Abstract

Kappa-Minkowski space-time is an example of noncommutative space-time with potentially interesting phenomenological consequences. However, the construction of field theories on this space, although operationally well-defined, is plagued with ambiguities. A part of ambiguities can be resolved by clarifying the geometrical picture of gauge transformations on the  $\kappa$ -Minkowski space-time. To this end we use the twist approach to construct the noncommutative  $U(1)$  gauge theory coupled to fermions. However, in this approach we cannot maintain the kappa-Poincaré symmetry; the corresponding symmetry of the twisted kappa-Minkowski space is the twisted  $\mathfrak{igl}(1,3)$  symmetry. We construct an action for the gauge and matter fields in a geometric way, as an integral of a maximal form. We use the Seiberg-Witten map to relate noncommutative and commutative degrees of freedom and expand the action to obtain the first order corrections in the deformation parameter.

# 1 Introduction

It is generally believed that the picture of space-time as a differentiable manifold should break down at very short distances of the order of the Planck length. There are different proposals for the modified space-time structure which should provide consistent framework encompassing physics in this regime. These proposals include, among others, the dynamical triangulation as a way of direct geometrical construction of modified space-time, strings and loops as non-local fundamental observables dynamically generating space-time, and a deformation of algebra of functions on a manifold as a way of introducing a 'noncommutative space-time'.

The theoretical motivation for introducing a non-trivial algebra of coordinates in order to modify the space-time structure comes from various ideas and results. Historically, the first proposal by Snyder [1] was put forward as a way of introducing a cut-off in quantum field theory, i.e., as a proposal for regularization of divergences. The quantum group approach [2] appeared as a generalisation of concept of symmetries that should encompass physics on a quantum manifold. Today, this idea is realised in the framework of spin foam models based on the representation theory of quantum groups [3]. More recently, the realisation that the open string theories and D-branes in the presence of a background antisymmetric B-field give rise to noncommutative effective field theories [4, 5] gave boost to research of field theories on a noncommutative space-time. It is believed that the field theories on noncommutative spaces taken as effective models are capable of capturing some generic features of an elusive quantum theory of gravity.

The main advantage of such effective models is that one can extract phenomenological consequences of the space-time modification using the standard field-theoretical tools. However, one needs to have a full understanding of the symmetry structure of these models and their renormalization properties to be able to give testable predictions.

In this work our primary interest is to examine compatibility of the local gauge principle with the deformation of algebra of functions on a specific example of noncommutative space-time, the  $\kappa$ -Minkowski space-time. The commutation relations of coordinates of the  $m$  dimensional  $\kappa$ -Minkowski space-time are of the Lie-algebra type

$$[\hat{x}^0, \hat{x}^j] = \frac{i}{\kappa} \hat{x}^j, \quad [\hat{x}^i, \hat{x}^j] = 0, \quad (1.1)$$

where  $i, j = 1, \dots, m-1$  and the zeroth component corresponds to the time direction. One of the interesting properties of this noncommutative space-time is that there is a quantum group symmetry acting on it. It is a dimensionfull deformation of the global Poincaré group, the  $\kappa$ -Poincaré group. The constant  $\kappa$  has dimension of energy and sets a deformation scale. Historically, the  $\kappa$ -Poincaré group was first obtained by Lukierski et al. in [6] by the Inönü-Wigner contraction of the  $q$ -anti-de Sitter Hopf algebra  $SO_q(3, 2)$ . The  $\kappa$ -Poincaré Hopf algebra was introduced in [7] as a dual symmetry structure to the  $\kappa$ -Poincaré group. The  $\kappa$ -Minkowski space-time is a module of this algebra. The  $\kappa$ -Poincaré group found its realisation in the Doubly Special

Relativity (DSR) theories [8]. These theories are introduced as a possible generalisation of Special Relativity with one additional invariant scale, usually taken to be an energy scale (of order of Planck energy) or a length scale (of order of Planck length), see discussion in [9]. The generalisation is done in such a way that it leads to a  $\kappa$ -Poincaré invariant modified dispersion relation for photons and the energy-dependent speed of light. However, the claim of [10] that this modification of the speed of light is 23 orders of magnitude stronger than the recent measurements of gamma-ray bursts, opened an intensive discussion, see [11]. One of the new ideas that originated from that discussion is the relativity of locality [12], an idea which should be relevant in the regime characterised by negligible  $\hbar$  and  $G$  (classical-non gravitational regime), so that both quantum and gravitational effects are small, with their ratio kept fixed. In this new approach, the modified dispersion relation is no longer invariant under the  $\kappa$ -Poincaré action, but it transforms when the reference frame is changed. This apparently leads to the modification of the speed of light that is of the same order as the expected measured corrections and therefore these two can be compared [13].

As we can see, the  $\kappa$ -Minkowski space-time is an example of noncommutative space-time with potentially interesting phenomenological consequences. The construction of field theories on this space, although operationally well-defined, is plagued with ambiguities. In our previous work [14] we showed that within the framework defined in [15, 16] one can consistently describe a gauge theory on the  $\kappa$ -Minkowski space-time by explicit construction of  $U(1)$  gauge theory coupled to fermions. Although successful, our construction revealed certain ambiguities which were fixed by the physical arguments and intuition, rather than by the formalism itself. We lacked a better understanding of the symmetries in the model and the geometrical formulation of gauge theory, with the gauge field viewed as the connection 1-form. With this motivation in mind, in this paper we use the twist formalism in order to gain a better understanding of the gauge theory on the  $\kappa$ -Minkowski space-time.

It is important to note that the twisted symmetry does not have the usual dynamical significance and there is no Noether procedure associated with it. In this paper we view this symmetry as a way of bookkeeping, a prescription that allow us to consistently apply deformation in the theory. The effective model obtained by expansion in the deformation parameter is however amenable to the usual analysis, and is the one from which one should draw out physical consequences of the deformation introduced.

In order for the paper to be self-consistent, in the next section we review some known results about the twisted differential geometry. In Section 3 we construct the  $\kappa$ -Minkowski space-time by choosing an explicit twist. Especially, we discuss the differential structure on the obtained space-time: differential calculus,  $\star$ -algebra of forms and integral. Using the mathematical tools introduced in the previous sections, in Section 4 the noncommutative  $U(1)$  gauge theory coupled to fermions is constructed. We use the Seiberg-Witten (SW) map to relate noncommutative and commutative degrees of freedom. The action for the gauge and matter fields is written in a geometric way, as an integral of a maximal form. We then expand the action up to first order in the deformation parameter, obtain equations of motions and discuss possible deformations

of dispersion relations for free fields. Finally, in Section 5 we discuss the obtained results and list some open questions and problems.

## 2 Noncommutative spaces from a twist

There are different ways to realize a noncommutative space and to formulate physical models on it, see [17] and [18]. One of the most discussed approaches is that of deformation quantization. In this approach a noncommutative space is a quotient of the algebra freely generated by the operators  $\hat{x}^\mu$  and divided by the ideal generated by the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}(\hat{x}) \quad (2.2)$$

where  $\Theta^{\mu\nu}(\hat{x})$  is an arbitrary polynomial of  $\hat{x}^\mu$  operators. This algebra can be represented on the space of commuting coordinates for the most interesting and/or the most studied examples<sup>1</sup>. The (noncommutative) algebra multiplication between two functions of noncommuting coordinates  $\hat{f}$  and  $\hat{g}$  is then mapped to the  $\star$ -product:

$$\hat{f} \cdot \hat{g}(\hat{x}) \mapsto f \star g(x) \in \mathcal{A}_x. \quad (2.3)$$

Here  $f$  and  $g$  are functions of the commuting coordinates and with  $\mathcal{A}_x$  we label the algebra of functions on the commutative space. The product (2.3) is bilinear and associative but noncommutative. The algebra of noncommuting coordinates  $\hat{\mathcal{A}}_{\hat{x}}$  is then isomorphic to the algebra of commuting coordinates with the  $\star$ -product (instead of the usual point-wise multiplication) as a multiplication. A well known example is the  $\star$ -product for the canonically deformed space defined by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (2.4)$$

where  $\theta^{\mu\nu}$  is an antisymmetric constant matrix of mass dimension minus two. The  $\star$ -product is given by the Moyal-Weyl product

$$\begin{aligned} f \star g(x) &= \lim_{x \rightarrow y} e^{\frac{i}{2}\theta^{\rho\sigma} \frac{\partial}{\partial x^\rho} \frac{\partial}{\partial y^\sigma}} f(x)g(y) \\ &= f \cdot g + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\rho_1\sigma_1} \dots \theta^{\rho_n\sigma_n} \left(\partial_{\rho_1} \dots \partial_{\rho_n} f(x)\right) \left(\partial_{\sigma_1} \dots \partial_{\sigma_n} g(x)\right). \end{aligned} \quad (2.5)$$

Now one can define a noncommutative space as the usual space of commuting coordinates with the point-wise multiplication replaced by a noncommutative  $\star$ -product. Different models were constructed using this approach. A noncommutative extension of the Standard Model was constructed in [20] and some phenomenological consequences were analyzed in [21]. Renormalization of different models was discussed in [22].

However, there is a drawback of this approach. Namely, it is not clear what happens with symmetries of the theory in this approach. For example, the commutation

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<sup>1</sup>The algebras need to fulfil the Poincaré-Birkhoff-Witt property [19], and this is true for the canonical deformation, the Lie-algebra type of deformation and the quantum group type of deformation.

relations (2.4) obviously break the global Lorentz symmetry, since  $\theta^{\mu\nu}$  is constant. Is there a deformed symmetry which replaces the global Lorentz symmetry in this case? If it exists, what is it? An answer to these question could be given using the twist formalism.

## 2.1 Deformation by a twist

The main idea of the twist formalism is to first deform the symmetry of the theory and then see the consequences this deformation has on the space-time itself. There is a well defined way to deform the symmetry Hopf algebra. In his paper [23] Drinfel'd introduced a notion of twist. The twist  $\mathcal{F}$  is an invertible operator which belongs to  $Ug \otimes Ug$ , where  $Ug$  is the universal enveloping algebra of the symmetry Lie algebra  $g$ . The universal enveloping algebra  $Ug$  is a Hopf algebra

$$\begin{aligned} [t^a, t^b] &= if^{abc}t^c, \\ \Delta(t^a) &= t^a \otimes 1 + 1 \otimes t^a, \\ \varepsilon(t^a) &= 0, \quad S(t^a) = -t^a. \end{aligned} \tag{2.6}$$

In the first line  $t^a$  label the generators of the symmetry algebra  $g$  and the structure constants are labelled by  $f^{abc}$ . In the second line the coproduct of the generator  $t^a$  is given. It encodes the Leibniz rule and specifies how the symmetry transformation acts on products of fields/representations. In the last line, the counit and the antipode are given. The properties which the twist  $\mathcal{F}$  has to satisfy are:

1. the cocycle condition

$$(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F}, \tag{2.7}$$

2. normalization

$$(id \otimes \epsilon)\mathcal{F} = (\epsilon \otimes id)\mathcal{F} = 1 \otimes 1, \tag{2.8}$$

3. perturbative expansion

$$\mathcal{F} = 1 \otimes 1 + \mathcal{O}(\lambda), \tag{2.9}$$

where  $\lambda$  is a small deformation parameter. The last property is not necessary. It provides an expansion around the undeformed case in the limit  $\lambda \rightarrow 0$ . We shall frequently use the notation (sum over  $\alpha = 1, 2, \dots, \infty$  is understood)

$$\mathcal{F} = f^\alpha \otimes f_\alpha, \quad \mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha, \tag{2.10}$$

where, for each value of  $\alpha$ ,  $\bar{f}^\alpha$  and  $\bar{f}_\alpha$  are two distinct elements of  $Ug$  (and similarly  $f^\alpha$  and  $f_\alpha$  are in  $Ug$ ).

We also introduce the universal  $\mathcal{R}$ -matrix

$$\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}, \tag{2.11}$$

where by definition  $\mathcal{F}_{21} = f_\alpha \otimes f^\alpha$ . In the sequel we use the notation

$$\mathcal{R} = R^\alpha \otimes R_\alpha, \quad \mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha. \tag{2.12}$$

## 2.2 Consequences of the twist

The twist acts on the symmetry Hopf algebra and gives the twisted symmetry Hopf algebra

$$\begin{aligned} [t^a, t^b] &= i f^{abc} t^c, \\ \Delta_{\mathcal{F}}(t^a) &= \mathcal{F} \Delta(t^a) \mathcal{F}^{-1} \\ \varepsilon(t^a) &= 0, \quad S_{\mathcal{F}}(t^a) = f^\alpha S(f_\alpha) S(t^a) S(\bar{f}^\beta) \bar{f}_\beta. \end{aligned} \quad (2.13)$$

We see that the algebra remains the same, while in general the comultiplication changes. This leads to the deformed Leibniz rule for the symmetry transformations when acting on product of fields.

We can now use the twist to deform the commutative geometry on space-time (vector fields, 1-forms, exterior algebra of forms, tensor algebra). The guiding principle is the observation that every time we have a bilinear map

$$\mu : X \times Y \rightarrow Z,$$

where  $X, Y, Z$  are vector spaces and when there is an action of the Lie algebra  $g$  (and therefore of  $\mathcal{F}^{-1}$ ) on  $X$  and  $Y$  we can combine the map  $\mu$  with the action of the twist. In this way we obtain the deformed map  $\mu_\star$ :

$$\mu_\star = \mu \mathcal{F}^{-1}. \quad (2.14)$$

The cocycle condition (2.7) implies that if  $\mu$  is an associative product then also  $\mu_\star$  is an associative product.

Let us analyze this deformation in more detail. For convenience we now consider one particular class of twists, the Abelian twists

$$\mathcal{F} = e^{-\frac{i}{2} \theta^{ab} X_a \otimes X_b}. \quad (2.15)$$

Here  $\theta^{ab}$  is a constant antisymmetric matrix,  $a, b = 1, 2, \dots, p \leq m$  and  $X_a = X_a^\mu \partial_\mu$  are commuting vector fields,  $[X_a, X_b] = 0$ . The algebra of vector fields on the space-time  $M$  we label with  $\Xi$  and the universal enveloping algebra of this algebra with  $U\Xi$ . Then  $\mathcal{F}$  belongs to  $U\Xi \otimes U\Xi$ . In the view of (2.6)-(2.9), the symmetry algebra is the algebra of diffeomorphisms generated by vector fields  $\xi = \xi^\mu \partial_\mu \in \Xi$ . Note that depending on the choice of vector fields  $X_a$  one can also consider a subalgebra of the diffeomorphism algebra such as Poincaré or conformal algebra.

Applying the inverse of the twist (2.15) to the usual point-wise multiplication of functions on the space-time  $M$ ,  $\mu(f \otimes g) = f \cdot g$ , we obtain the  $\star$ -product of functions

$$\begin{aligned} f \star g &= \mu \mathcal{F}^{-1}(f \otimes g) \\ &= \bar{f}^\alpha(f) \bar{f}_\alpha(g) \\ &= \bar{R}^\alpha(g) \star \bar{R}_\alpha(f). \end{aligned} \quad (2.16)$$

We see that the  $R$ -matrix encodes the noncommutativity of the  $\star$ -product. The action of the twist ( $\bar{f}^\alpha$  and  $\bar{f}_\alpha$ ) on the functions  $f$  and  $g$  is via the Lie derivative.

The product between functions and 1-forms is given again by following the general prescription

$$h \star \omega = \bar{f}^\alpha(h) \bar{f}_\alpha(\omega) \quad (2.17)$$

with an arbitrary 1-form  $\omega$ . The action of  $\bar{f}_\alpha$  on forms is (again) given via the Lie derivative. Functions can be multiplied from the left or from the right,

$$h \star \omega = \bar{f}^\alpha(h) \bar{f}_\alpha(\omega) = \bar{R}^\alpha(\omega) \star \bar{R}_\alpha(h). \quad (2.18)$$

Exterior forms form an algebra with the wedge product  $\wedge : \Omega^\bullet \times \Omega^\bullet \rightarrow \Omega^\bullet$ . We  $\star$ -deform the wedge product on two arbitrary forms  $\omega$  and  $\omega'$  into the  $\star$ -wedge product,

$$\omega \wedge_\star \omega' = \bar{f}^\alpha(\omega) \wedge \bar{f}_\alpha(\omega'). \quad (2.19)$$

We denote by  $\Omega_\star^\bullet$  the linear space of forms equipped with the  $\star$ -wedge product  $\wedge_\star$ .

As in the commutative case, the exterior forms are totally  $\star$ -antisymmetric (contravariant) tensor-fields. For example, the 2-form  $\omega \wedge_\star \omega'$  is the  $\star$ -antisymmetric combination

$$\begin{aligned} \omega \wedge_\star \omega' &= \bar{f}^\alpha(\omega) \wedge \bar{f}_\alpha(\omega') \\ &= \omega \otimes_\star \omega' - \bar{R}^\alpha(\omega') \otimes_\star \bar{R}_\alpha(\omega), \\ &= -\bar{R}^\alpha(\omega') \wedge_\star \bar{R}_\alpha(\omega), \end{aligned} \quad (2.20)$$

with the  $\star$ -tensor product defined as

$$T_1 \otimes_\star T_2 = \bar{f}^\alpha(T_1) \otimes \bar{f}_\alpha(T_2). \quad (2.21)$$

The usual exterior derivative  $d : \mathcal{A}_x \rightarrow \Omega$  satisfies the Leibniz rule<sup>2</sup>  $d(f \star g) = df \star g + f \star dg$  and is therefore also the  $\star$ -exterior derivative. One can rewrite the usual exterior derivative of a function using the  $\star$ -product as

$$\begin{aligned} df &= (\partial_\mu f) dx^\mu \\ &= (\partial_\mu^\star f) \star dx^\mu, \end{aligned} \quad (2.22)$$

where the new derivatives  $\partial_\mu^\star$  are defined by this equation.

The usual integral is cyclic under the  $\star$ -exterior products of forms, that is up to boundary terms we have

$$\int \omega_1 \wedge_\star \omega_2 = (-1)^{d_1 \cdot d_2} \int \omega_2 \wedge_\star \omega_1, \quad (2.23)$$

where  $d = \deg(\omega)$ ,  $d_1 + d_2 = m$  and  $m$  is the dimension of the space-time  $M$ . This property holds for the Abelian twist (2.15). More generally, one can show [24] that this property holds for any twist that satisfies the condition  $S(\bar{f}^\alpha) \bar{f}_\alpha = 1$ , with the antipode  $S$ .

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<sup>2</sup>The reason for this is that the usual exterior derivative commutes with the Lie derivative.

### 3 Kappa-Minkowski via twist

Algebraically, the  $m$ -dimensional  $\kappa$ -Minkowski space-time can be introduced as a quotient of the algebra freely generated by the coordinates  $\hat{x}^\mu$  and divided by the ideal generated by the following commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = iC_\rho^{\mu\nu} \hat{x}^\rho, \quad \mu, \nu, \rho = 0, \dots, m-1. \quad (3.24)$$

Defining

$$C_\rho^{\mu\nu} = a(\delta_0^\mu \delta_\rho^\nu - \delta_0^\nu \delta_\rho^\mu) \quad (3.25)$$

the commutation relations (3.24) can be rewritten as

$$[\hat{x}^0, \hat{x}^j] = ia\hat{x}^j, \quad [\hat{x}^i, \hat{x}^j] = 0. \quad (3.26)$$

The metric of the  $\kappa$ -Minkowski space-time is  $\eta^{\mu\nu} = \text{diag}(1, -1, \dots, -1)$ . The deformation parameter  $a$  is related to the frequently used parameter  $\kappa$  as  $a = 1/\kappa$ . Latin indices denote space dimensions, zero the time dimension and the Greek indices refer to all  $m$  dimensions.

As we have said in the previous section there exists an isomorphism between the abstract algebra and the algebra of functions of commuting coordinates equipped with a  $\star$ -product. There are different  $\star$ -product realizations of the  $\kappa$ -Minkowski space-time, see [25]. The symmetric  $\star$ -product for the  $\kappa$ -Minkowski space-time, up to the first order in the deformation parameter, is given by

$$\begin{aligned} f(x) \star_{SO} g(x) &= f(x)g(x) + \frac{i}{2} C_\lambda^{\mu\nu} x^\lambda \partial_\mu f(x) \partial_\nu g(x) \\ &= f(x)g(x) + \frac{ia}{2} x^j (\partial_0 f(x) \partial_j g(x) - \partial_j f(x) \partial_0 g(x)). \end{aligned} \quad (3.27)$$

Using this  $\star$ -product, field theories on  $\kappa$ -Minkowski were constructed [26, 15]. However, there are some open problems in this approach. We mention two of them. The ordinary partial derivative  $\partial_\mu = \frac{\partial}{\partial x^\mu}$  has a deformed Leibniz rule due to the  $x$ -dependence of the  $\star_{SO}$ -product,

$$\partial_\mu (f \star_{SO} g) = (\partial_\mu f) \star_{SO} g + f \star_{SO} (\partial_\mu g) + f(\partial_\mu \star_{SO})g. \quad (3.28)$$

This property can lead to gauge fields and field strength given in terms of higher order differential operators [14]. From the algebraic point of view there are different choices of derivatives and one has to specify a criterion (e.g., transformation under  $\kappa$ -Lorentz symmetry, convenient Leibniz rule) for choosing one particular set<sup>3</sup>. The definition of an integral is also a problem. The usual integral is not cyclic (again due to the  $x$ -dependence of the  $\star$ -product) and one has to introduce the measure function  $\mu$  in order to make it cyclic<sup>4</sup>,

$$\int d^m x \mu(x) f \star_{SO} g = \int d^m x \mu(x) g \star_{SO} f. \quad (3.29)$$

<sup>3</sup>For different approaches to the problem of differential calculus on  $\kappa$ -Minkowski space-time see [27].

<sup>4</sup>Scalar field theory with a cyclic integral and without a measure function was constructed and discussed in [28].



The equality holds up to boundary terms. In general, the measure function spoils the symmetry properties of an action and of the corresponding equations of motion. It also spoils the commutative limit since it is  $a$ -independent and does not vanish in the limit  $a \rightarrow 0$ .

In order to overcome some of these problems in this paper we follow the twist approach. The choice of twist is not unique and it depends on the properties that we want to obtain/preserve. We choose the following twist

$$\begin{aligned}\mathcal{F} &= e^{-\frac{i}{2}\theta^{ab}X_a \otimes X_b} \\ &= e^{-\frac{ia}{2}(\partial_0 \otimes x^j \partial_j - x^j \partial_j \otimes \partial_0)},\end{aligned}\tag{3.30}$$

with two commuting vector fields  $X_1 = \partial_0$  and  $X_2 = x^j \partial_j$  and

$$\theta^{ab} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}.\tag{3.31}$$

This twist fulfils the conditions (2.7), (2.8) and (2.9) with the small deformation parameter  $\lambda = a$ . Deformed symmetry concerned, note that  $X_2$  is not in the universal enveloping algebra of the Poincaré algebra. Therefore we have to enlarge the Poincaré algebra  $iso(1, m-1)$  to the inhomogeneous general linear algebra  $igl(1, m-1)$  and twist this algebra instead of  $iso(1, m-1)$ . The generators (given in the representation on the space of functions/fields) and the commutation relations of  $igl(1, m-1)$  are

$$\begin{aligned}M_{\mu\nu} &= x_\mu \partial_\nu, \quad P_\mu = \partial_\mu, \\ [P_\mu, P_\nu] &= 0, \quad [M_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\sigma} M_{\rho\nu}.\end{aligned}\tag{3.32}$$

Let us discuss the consequences of the twist (3.30).

### 3.1 Twisted symmetry

The action of the twist (3.30) on the  $igl(1, m-1)$  algebra follows from (2.13) and it has been analysed in detail in [29]. Let us just summarise the most important results. The algebra (3.32) remains the same. On the other hand, since  $X_2 = x^j \partial_j$  does not commute with the generators  $P_\mu$  and  $M_{\mu\nu}$  the comultiplication and the antipode change. Here we just give the result for the twisted comultiplication, the other results can be found in [29].

$$\begin{aligned}\Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta P_j &= P_j \otimes e^{-\frac{i}{2}aP_0} + e^{\frac{i}{2}aP_0} \otimes P_j, \\ \Delta M_{ij} &= M_{ij} \otimes 1 + 1 \otimes M_{ij}, \\ \Delta M_{0j} &= M_{0j} \otimes e^{-\frac{i}{2}aP_0} + e^{\frac{i}{2}aP_0} \otimes M_{0j} - \frac{i}{2}aP_j \otimes D + \frac{i}{2}aD \otimes P_j, \\ \Delta M_{j0} &= M_{j0} \otimes e^{-\frac{i}{2}aP_0} + e^{\frac{i}{2}aP_0} \otimes M_{j0}, \\ \Delta M_{00} &= M_{00} \otimes 1 + 1 \otimes M_{00} - \frac{i}{2}aP_0 \otimes D + \frac{i}{2}aD \otimes P_0.\end{aligned}\tag{3.33}$$

We introduced the notation  $D = x^j \partial_j$ . Note that  $\kappa$ -Poincaré symmetry found in [6] will not be a symmetry of our twisted  $\kappa$ -Minkowski space. The corresponding symmetry of the twisted  $\kappa$ -Minkowski space is the twisted  $igl(1, m-1)$  symmetry.

### 3.2 $\star$ -product

The inverse of the twist (3.30) defines the  $\star$ -product between functions/fields on the  $\kappa$ -Minkowski space-time

$$\begin{aligned} f \star g &= \mu_\star \{f \otimes g\} \\ &= \mu \{ \mathcal{F}^{-1} f \otimes g \} \\ &= \mu \{ e^{\frac{ia}{2}(\partial_0 \otimes x^j \partial_j - x^j \partial_j \otimes \partial_0)} f \otimes g \} \end{aligned} \quad (3.34)$$

$$\begin{aligned} &= f \cdot g + \frac{ia}{2} x^j ((\partial_0 f) \partial_j g - (\partial_j f) \partial_0 g) + \mathcal{O}(a^2) \\ &= f \cdot g + \frac{i}{2} C_\lambda^{\rho\sigma} x^\lambda (\partial_\rho f) \cdot (\partial_\sigma g) + \mathcal{O}(a^2), \end{aligned} \quad (3.35)$$

with  $C_\lambda^{\rho\sigma}$  given in (3.25) This product is associative, noncommutative and hermitean

$$\overline{f \star g} = \bar{g} \star \bar{f}. \quad (3.36)$$

The usual complex conjugation we label with “bar”. In the zeroth order (3.35) reduces to the usual point-wise multiplication. Note that the first order term of this  $\star$ -product is the same as the first order term of the symmetric  $\star_{SO}$ -product (3.27). The second and higher orders will be different. Of course, we obtain

$$[x^0 \star, x^j] = x^0 \star x^j - x^j \star x^0 = i a x^j, \quad [x^i \star, x^j] = 0. \quad (3.37)$$

### 3.3 Twisted differential calculus

One of the advantages of the twist formalism is the straightforward way to define a differential calculus. Namely, as said in the previous section, we just adopt the undeformed differential calculus with the following properties

$$\begin{aligned} d(f \star g) &= df \star g + f \star dg, \\ d^2 &= 0, \\ df &= (\partial_\mu f) dx^\mu = (\partial_\mu^\star f) \star dx^\mu. \end{aligned} \quad (3.38)$$

The basis one forms are  $dx^\mu$ . Knowing that the action of a vector field on a form is given via Lie derivative one can show that

$$X_1(dx^\mu) = 0, \quad X_2(dx^\mu) = \delta_j^\mu dx^j. \quad (3.39)$$

Using these relations one obtains that the basis 1-forms anticommute but do not  $\star$ -commute with functions. They are not frame 1-forms in the sense of Madore [17]. Instead they fulfil

$$\begin{aligned} dx^\mu \wedge_\star dx^\nu &= dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu = -dx^\nu \wedge_\star dx^\mu, \\ f \star dx^0 &= dx^0 \star f, \quad f \star dx^j = dx^j \star e^{ia\partial_0} f. \end{aligned} \quad (3.40)$$

Arbitrary 1-forms  $\omega_1 = \omega_{1\mu} \star dx^\mu$  and  $\omega_2 = \omega_{2\mu} \star dx^\mu$  do not anticommute

$$\omega_1 \wedge_\star \omega_2 = -\bar{R}^\alpha(\omega_2) \wedge_\star \bar{R}_\alpha(\omega_1), \quad (3.41)$$

where the inverse of the  $\mathcal{R}$  matrix is given by

$$\mathcal{R}^{-1} = \mathcal{F}^2 = e^{-ia(\partial_0 \otimes x^j \partial_j - x^j \partial_j \otimes \partial_0)}. \quad (3.42)$$

The  $\star$ -derivatives follow from (3.38) and are given by

$$\begin{aligned} \partial_0^\star &= \partial_0, \quad \partial_j^\star = e^{-\frac{i}{2}a\partial_0} \partial_j, \\ \partial_0^\star(f \star g) &= (\partial_0^\star f) \star g + f \star (\partial_0^\star g), \\ \partial_j^\star(f \star g) &= (\partial_j^\star f) \star e^{-ia\partial_0} g + f \star (\partial_j^\star g). \end{aligned} \quad (3.43)$$

### 3.4 Integral

The usual integral of a maximal form is cyclic

$$\int \omega_1 \wedge_\star \omega_2 = (-1)^{d_1 \cdot d_2} \int \omega_2 \wedge_\star \omega_1, \quad (3.44)$$

with  $d = \deg(\omega)$  and  $d_1 + d_2 = m$ . Since basis 1-forms anticommute the volume form remains undeformed

$$d_\star^m x := dx^0 \wedge_\star dx^1 \wedge_\star \dots dx^{m-1} = dx^0 \wedge dx^1 \wedge \dots dx^{m-1} = d^m x. \quad (3.45)$$

## 4 $U(1)$ gauge theory

In this section we formulate a noncommutative  $U(1)$  gauge theory coupled to fermionic matter. We follow the Seiberg-Witten method [5] and the enveloping algebra approach [30]. From now on we work in four dimensions. The method is however general, it can be applied to any  $SU(N)$  and  $U(N)$  gauge group and in any number of dimensions.

The basic assumption of the SW map is that the noncommutative fields and the noncommutative gauge parameter can be expressed as functions of the commutative fields and the commutative gauge parameter  $\alpha$ . For example, the noncommutative gauge parameter  $\Lambda$  is

$$\Lambda = \Lambda(\alpha, A_\mu^0) := \Lambda_\alpha(A_\mu^0) \quad (4.46)$$

with the commutative<sup>5</sup> gauge field  $A_\mu^0$ . The explicit form of this dependence is found by solving the appropriate equations. In that way the number of degrees of freedom in the noncommutative theory reduces to the number of degrees of freedom of the corresponding commutative theory.

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<sup>5</sup>In the following, superscript zero denotes undeformed, commutative fields.

## 4.1 Matter fields

The infinitesimal noncommutative gauge transformation of the field  $\psi$  is given by

$$\delta_\alpha^\star \psi = i\Lambda_\alpha \star \psi, \quad (4.47)$$

where  $\Lambda_\alpha$  is the noncommutative gauge parameter related via SW map with the commutative gauge parameter  $\alpha$  and  $\psi$  is the noncommutative matter field in the fundamental representation. We demand consistency, that is that the algebra of gauge transformation closes:

$$(\delta_\alpha^\star \delta_\beta^\star - \delta_\beta^\star \delta_\alpha^\star) \psi = \delta_{-i[\alpha, \beta]}^\star \psi. \quad (4.48)$$

In order to solve this equation, we expand  $\Lambda_\alpha$  in the orders of the deformation parameter

$$\Lambda_\alpha = \alpha + \Lambda_\alpha^1 + \dots + \Lambda_\alpha^k + \dots \quad (4.49)$$

Also we have to expand the  $\star$ -product in the equation (4.48). All the expansions in this paper will be up to first order in the deformation parameter  $a$ . The inhomogeneous equation for  $\Lambda_\alpha^1$  then reads

$$\begin{aligned} \delta_\alpha \Lambda_\beta^1 - \delta_\beta \Lambda_\alpha^1 - i[\alpha, \Lambda_\beta^1] - i[\Lambda_\alpha^1, \beta] - \Lambda_{-i[\alpha, \beta]}^1 &= -\frac{1}{2} C_\lambda^{\rho\sigma} x^\lambda \{\partial_\rho \alpha, \partial_\sigma \beta\}, \\ \delta_\alpha \Lambda_\beta^1 - \delta_\beta \Lambda_\alpha^1 &= -C_\lambda^{\rho\sigma} x^\lambda (\partial_\rho \alpha) (\partial_\sigma \beta). \end{aligned} \quad (4.50)$$

In the second line we used the fact that in the case of  $U(1)$  gauge symmetry all commutators vanish and all anticommutators just add. Note that  $\delta_\alpha \Lambda_\beta^1 \neq 0$  since  $\Lambda_\beta^1$  is a function of the commutative gauge parameter  $\beta$  and the commutative gauge field  $A_\mu^0$  and  $\delta_\alpha A_\mu^0 = \partial_\mu \alpha \neq 0$ . The solution of equation (4.50) is given by

$$\Lambda_\alpha^1 = -\frac{1}{2} C_\lambda^{\rho\sigma} x^\lambda A_\rho^0 \partial_\sigma \alpha. \quad (4.51)$$

This solution is not unique, one can always add a solution of the homogeneous equation to it. This is the freedom in the SW map. In the case of  $U(1)$  gauge group the only homogeneous term is of the form

$$\Lambda_\alpha^{\text{hom}} = c_1 C_\lambda^{\rho\sigma} x^\lambda F_{\rho\sigma}^0 \alpha, \quad (4.52)$$

with the commutative field-strength tensor  $F_{\rho\sigma}^0 = \partial_\rho A_\sigma^0 - \partial_\sigma A_\rho^0$ . However, this term does not lead to a solvable equation for the noncommutative gauge field and therefore we shall not consider it. The noncommutative gauge parameter up to first order in the deformation parameter reads

$$\Lambda_\alpha = \alpha - \frac{1}{2} C_\lambda^{\mu\nu} x^\lambda A_\mu^0 \partial_\nu \alpha. \quad (4.53)$$

The solution for the matter field  $\psi$  follows from (4.47) and (4.53) and is given by

$$\psi = \psi^0 - \frac{1}{2} C_\lambda^{\rho\sigma} x^\lambda A_\rho^0 (\partial_\sigma \psi^0) + i d_1 C_\lambda^{\rho\sigma} x^\lambda F_{\rho\sigma}^0 \psi^0 + d_2 a D_0^0 \psi^0. \quad (4.54)$$

The terms with the real undetermined coefficients  $d_1$  and  $d_2$  are the solutions of the homogeneous equation and represent the freedom of the SW map.

## 4.2 Gauge fields

In order to write a gauge invariant action for the matter field  $\psi$  one has to introduce a covariant derivative and a connection. We have a preferred differential calculus on the  $\kappa$ -Minkowski space-time given by (3.38) and we will use it now. The covariant derivative  $D\psi$  is defined in the following way

$$D\psi = d\psi - iA \star \psi = D_\mu^\star \psi \star dx^\mu, \quad (4.55)$$

$$D_0^\star \psi = \partial_0^\star \psi - iA_0 \star \psi, \quad D_j^\star \psi = \partial_j^\star \psi - iA_j \star e^{-ia\partial_0} \psi, \quad (4.56)$$

where the noncommutative connection  $A = A_\mu \star dx^\mu$  is introduced. The term  $e^{-ia\partial_0} \psi$  comes from  $\star$ -commuting  $\psi$  through  $dx^j$ . The transformation law of the covariant derivative

$$\delta_\alpha^\star D\psi = i\Lambda_\alpha \star D\psi \quad (4.57)$$

defines the transformation law of the noncommutative connection. It is given by

$$\delta_\alpha^\star A = d\Lambda_\alpha + i[\Lambda_\alpha \star A], \quad (4.58)$$

or in the components

$$\delta_\alpha^\star A_0 = \partial_0 \Lambda_\alpha + i[\Lambda_\alpha \star A_0], \quad (4.59)$$

$$\delta_\alpha^\star A_j = \partial_j^\star \Lambda_\alpha + i\Lambda_\alpha \star A_j - iA_j \star e^{-ia\partial_0} \Lambda_\alpha. \quad (4.60)$$

Assuming that  $A_\mu = A_\mu^0 + A_\mu^1 + \dots$  one finds the solutions of (4.59) and (4.60)

$$\begin{aligned} A_\mu = & A_\mu^0 - \frac{a}{2} \delta_\mu^j \left( i\partial_0 A_j^0 + A_0^0 A_j^0 \right) + \frac{1}{2} C_\lambda^{\rho\sigma} x^\lambda \left( F_{\rho\mu}^0 A_\sigma^0 - A_\rho^0 \partial_\sigma A_\mu^0 \right) \\ & + d_3 C_\lambda^{\rho\sigma} x^\lambda \partial_\rho F_{\sigma\mu}^0 + d_4 a F_{\mu 0}. \end{aligned} \quad (4.61)$$

The terms with the real, undetermined coefficients  $d_3$  and  $d_4$  are the solutions of the homogeneous equation and represent the freedom of the SW map. Note that the connection 1-form  $A$  is real, but the components  $A_\mu$  are not necessarily real due to the  $\star$ -product in  $A = A_\mu \star dx^\mu$ .

In the next step we construct the field-strength tensor. The field-strength tensor is a two-form given by<sup>6</sup>

$$F = \frac{1}{2} F_{\mu\nu} \star dx^\mu \wedge_\star dx^\nu = dA - iA \wedge_\star A, \quad (4.62)$$

or in components

$$F_{0j} = \partial_0^\star A_j - \partial_j^\star A_0 - iA_0 \star A_j + iA_j \star e^{-ia\partial_0} A_0, \quad (4.63)$$

$$F_{ij} = \partial_i^\star A_j - \partial_j^\star A_i - iA_i \star e^{-ia\partial_0} A_j + iA_j \star e^{-ia\partial_0} A_i. \quad (4.64)$$

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<sup>6</sup>Equivalently, one can define the field-strength tensor as  $D^2\psi = -iF \star \psi$ , with the covariant derivative  $D$  given in (4.55).

One can check that the field-strength tensor transforms covariantly,

$$\delta_\alpha^* F = i[\Lambda_\alpha \star F]. \quad (4.65)$$

Inserting the solution (4.61) into (4.63) and (4.64) results in

$$\begin{aligned} F_{0j} &= F_{0j}^0 - \frac{ia}{2} \partial_0 F_{0j}^0 - a A_0^0 F_{0j}^0 + C_\lambda^{\rho\sigma} x^\lambda (F_{\rho 0}^0 F_{\sigma j}^0 - A_\rho^0 \partial_\sigma F_{0j}^0) \\ &\quad + a(d_3 - d_4) \partial_0 F_{0j}^0, \end{aligned} \quad (4.66)$$

$$\begin{aligned} F_{ij} &= F_{ij}^0 - ia \partial_0 F_{ij}^0 - 2a A_0^0 F_{ij}^0 + C_\lambda^{\rho\sigma} x^\lambda (F_{\rho i}^0 F_{\sigma j}^0 - A_\rho^0 \partial_\sigma F_{ij}^0) \\ &\quad + a(d_3 - d_4) \partial_0 F_{ij}^0. \end{aligned} \quad (4.67)$$

### 4.3 Gauge field action

In the commutative gauge theory one writes the action for the gauge field using the Hodge dual of the field-strength tensor,  $*F^0$ :

$$\begin{aligned} S_g^0 &= \int F^0 \wedge (*F^0), \\ *F^0 &= \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{0\alpha\beta} dx^\mu \wedge dx^\nu. \end{aligned}$$

The indices on  $F^{0\alpha\beta}$  are raised with the flat metric  $\eta_{\mu\nu}$  and

$$\delta_\alpha(*F^0) = i[\alpha, *F^0] = 0 \quad (4.68)$$

since we work with  $U(1)$  gauge theory.

We try to generalise this to the  $\kappa$ -Minkowski space-time. We write the noncommutative gauge field action as

$$S = c_1 \int F \wedge_\star (*F), \quad (4.69)$$

where  $*F$  is the noncommutative dual field-strength tensor. In order to have an action invariant under the noncommutative gauge transformations (4.58), tensor  $*F$  has to transform covariantly

$$\delta_\alpha^*(*F) = i[\Lambda_\alpha \star *F]. \quad (4.70)$$

The obvious guess for the noncommutative Hodge dual

$$*F = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \star dx^\mu \wedge_\star dx^\nu \quad (4.71)$$

does not work since it does not transform covariantly

$$\delta_\alpha^*(*F) = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} (\delta_\alpha^* F^{\alpha\beta}) \star dx^\mu \wedge_\star dx^\nu \neq i[\Lambda_\alpha \star *F]. \quad (4.72)$$

Therefore we have to try something else. We assume that  $*F$  has the form

$$*F := \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} X^{\alpha\beta} \star dx^\mu \wedge_\star dx^\nu, \quad (4.73)$$

where  $X^{\alpha\beta}$  are unknown components that should be determined from the condition (4.70). One way to determine these components is by using the SW map and assuming that

$$X^{\alpha\beta} = F^{0\alpha\beta} + X^{1\alpha\beta} + \dots \quad (4.74)$$

This would provide us with tensor  $X^{\alpha\beta}$  as a function of the commutative field  $A_\mu^0$ . The other possibility is to make an Ansatz, consistent with (4.70), on the functional dependence of  $X^{\alpha\beta}$  on the noncommutative field  $A_\mu$ . When expanded in the deformation parameter, both possibilities should give the same dual field strength. However, the first approach would generate additional ambiguous term in  $*F$  coming from the freedom in SW map for  $X^{\alpha\beta}$ , while in the second approach  $*F$  "inherits" the ambiguities from the SW map for  $A_\mu$ . Wishing to keep the ambiguities under control, we choose the second approach. Unfortunately, we were unable to find consistent Ansatz for  $X^{\alpha\beta}(A_\mu)$  in the closed form. Up to the first order in the deformation parameter we find:

$$\begin{aligned} X^{0j} &= F^{0j} - aA_0 \star F^{0j}, \\ X^{jk} &= F^{jk} + aA_0 \star F^{jk}. \end{aligned} \quad (4.75)$$

Inserting this into (4.73) gives dual field strength that does transform covariantly under the gauge transformations.

Going back to the action (4.69) and writing it more explicitly we obtain

$$S_g = -\frac{1}{4} \int \left\{ 2F_{0j} \star e^{-ia\partial_0} X^{0j} + F_{ij} \star e^{-2ia\partial_0} X^{ij} \right\} \star d^4x. \quad (4.76)$$

where the components of  $F$  and  $X$  are given in (4.63), (4.64) and (4.75). The terms  $e^{-ia\partial_0} X^{0j}$  and  $e^{-2ia\partial_0} X^{ij}$  come from  $\star$ -commuting basis 1-forms with the components  $X^{\mu\nu}$ . The constant  $c_1$  is fixed in such a way as to give the good commutative limit of the action (4.76).

#### 4.4 Matter field action

There are different ways to write a noncommutative gauge invariant action for spinor matter fields. Since we want to use the cyclicity property of the integral (3.44) we have to write the action as an integral of a maximal form. To this end we introduce the vierbein 1-forms

$$V = V_\mu \star dx^\mu = V_\mu^a \gamma_a \star dx^\mu, \quad (4.77)$$

with the Dirac gamma matrices in four dimensions  $\gamma_a$  and  $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$ . Since we work in the flat space-time  $V_\mu^a = \delta_\mu^a$  and the vierbeins (4.77) reduce to

$$V = \gamma_\mu \star dx^\mu = \gamma_\mu dx^\mu. \quad (4.78)$$

Noncommutative gauge invariance implies the following action

$$S_m = c_2 \int \left( (\overline{D\psi})_B \star \psi_A - \bar{\psi}_B \star (D\psi)_A \right) \wedge_\star (V \wedge_\star V \wedge_\star V \gamma_5)_{BA}, \quad (4.79)$$

with spinor indices  $A, B$  explicitly written (see Appendix for the explicit calculation) and  $\overline{D\psi} = dx^\mu \star \overline{D_\mu^\star \psi}$ . Knowing that under the noncommutative infinitesimal gauge transformations

$$\delta_\alpha^\star V = 0, \quad \delta_\alpha^\star \psi = i\Lambda_\alpha \star \psi, \quad \delta_\alpha^\star \bar{\psi} = -i\bar{\psi} \star \Lambda_\alpha \quad (4.80)$$

one can explicitly show that the action (4.79) is gauge invariant. In the commutative limit  $a \rightarrow 0$  this action reduces to the commutative action for spinor fields [24]

$$S_m^0 = \frac{1}{2} \text{Tr} \int \left( (D\psi)\bar{\psi} - \psi(\overline{D\psi}) \right) \wedge (V \wedge V \wedge V \gamma_5). \quad (4.81)$$

In order to write the action (4.79) in a form more convenient for calculating equations of motion we have to calculate the trace over spinor indices. We do this calculation explicitly in Appendix and give the result here:

$$\begin{aligned} S_m &= c_2 \int \left( (\overline{D\psi})_B \star \psi_A - \bar{\psi}_B \star (D\psi)_A \right) \wedge_\star (V \wedge_\star V \wedge_\star V \gamma_5)_{BA} \\ &= 6c_2 \int \left( i(\overline{D_\mu^\star \psi}) \gamma^\mu \star \psi - \bar{\psi} \star (i\gamma^\mu D_\mu^\star \psi) \right) \star d^4x \end{aligned}$$

with

$$\begin{aligned} D_0^\star \psi &= \partial_0^\star \psi - iA_0 \star \psi, & D_j^\star \psi &= \partial_j^\star \psi - iA_j \star e^{-ia\partial_0} \psi, \\ \overline{D_0^\star \psi} &= \partial_0^\star \bar{\psi} + i\bar{\psi} \star A_0, & \overline{D_j^\star \psi} &= \overline{\partial_j^\star \psi} + ie^{ia\partial_0} (\bar{\psi} \star A_j). \end{aligned} \quad (4.82)$$

As we already stressed, the twist (3.30) leads to the twisted  $igl(1,3)$  symmetry. Since the  $igl(1,3)$  algebra contains the conformal subalgebra, introducing a mass term for the fermions would break the conformal symmetry<sup>7</sup> and therefore the full  $igl(1,3)$ . Therefore, the twisted  $igl(1,3)$  invariant and the gauge invariant action for the spinor matter field  $\psi$  reads:

$$S_m = \frac{1}{2} \int \left( \bar{\psi} \star i\gamma^\mu (D_\mu^\star \psi) - i\overline{D_\mu^\star \psi} \gamma^\mu \star \psi \right) \star d^4x. \quad (4.83)$$

Notice that we (again) adjusted  $c_2$  to get the good commutative limit.

## 4.5 Equations of motion

Having defined the action in the previous subsections, we are ready to calculate the equations of motion for the fields. As we are interested only in the first order corrections in the noncommutativity parameter  $a$ , we can proceed following two different procedures. We can vary the complete action  $S = S_g + S_m$ , defined by equations (4.76) and (4.83), with respect to the noncommutative fields and then expand those equation and use the SW map to obtain the corresponding equations of motion for the

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<sup>7</sup>We would like to thank R.T. Govindarajan for drawing our attention to this point.



commutative fields with the first order corrections. Alternatively, we can first expand the complete action up to the first order in  $a$ , use the SW map, and then vary thus obtained action with respect to the commutative degrees of freedom. Both procedures give equivalent equations of motion for the commutative degrees of freedom with the first order corrections. Here we present the second option, writing explicitly the expanded action for the commutative degrees of freedom and the corresponding equations of motion.

The expanded action reads

$$\begin{aligned} S^{\text{expand}} &= S_g^{\text{expand}} + S_m^{\text{expand}}, \\ S_g^{\text{expand}} &= -\frac{1}{4} \int d^4x \left\{ F_{\mu\nu}^0 F^{0\mu\nu} - \frac{1}{2} C_\lambda^{\rho\sigma} x^\lambda F^{0\mu\nu} F_{\mu\nu}^0 F_{\rho\sigma}^0 + \right. \\ &\quad \left. + 2C_\lambda^{\rho\sigma} x^\lambda F^{0\mu\nu} F_{\mu\rho}^0 F_{\nu\sigma}^0 \right\}, \end{aligned} \quad (4.84)$$

$$\begin{aligned} S_m^{\text{expand}} &= \frac{1}{2} \int d^4x \left\{ \bar{\psi}^0 \left( i\gamma^\mu D_\mu^0 \psi^0 + \frac{a}{2} \gamma^j D_0^0 D_j^0 \psi^0 + \frac{i}{2} C_\lambda^{\rho\sigma} x^\lambda \gamma^\mu F_{\rho\mu}^0 (D_\sigma \psi^0) \right) \right. \\ &\quad \left. - \left( i\overline{D_\mu \psi^0} \gamma^\mu - \frac{a}{2} \overline{D_0 D_j \psi^0} \gamma^j + \frac{i}{2} C_\lambda^{\rho\sigma} x^\lambda \overline{D_\sigma \psi^0} \gamma^\mu F_{\rho\mu}^0 \right) \psi^0 \right\}. \end{aligned} \quad (4.85)$$

Note that there are no ambiguous terms in the expanded action coming from the freedom in SW map; all such terms turned out to be total derivative terms and therefore they dropped out from the expanded action. The equations for fermions are:

$$\begin{aligned} i\gamma^\mu D_\mu^0 \psi^0 + \frac{a}{2} \gamma^j D_0^0 D_j^0 \psi^0 + \frac{i}{2} C_\lambda^{\rho\sigma} x^\lambda \gamma^\mu F_{\rho\mu}^0 (D_\sigma \psi^0) &= 0, \\ -i\overline{D_\mu \psi^0} \gamma^\mu + \frac{a}{2} \overline{D_0 D_j \psi^0} \gamma^j - \frac{i}{2} C_\lambda^{\rho\sigma} x^\lambda \overline{D_\sigma \psi^0} \gamma^\mu F_{\rho\mu}^0 &= 0, \end{aligned} \quad (4.86)$$

while for the gauge field we obtain:

$$\begin{aligned} \partial_\mu F^{0\alpha\mu} + \frac{a}{4} \delta_0^\alpha F^{0\mu\nu} F_{\mu\nu}^0 + 2a F^{0\alpha\mu} F_{0\mu}^0 - C_\lambda^{\rho\sigma} x^\lambda \left( \partial_\mu (F_\rho^{0\mu} F_\sigma^{0\alpha}) + F_{\mu\sigma}^0 (\partial_\rho F^{0\mu\alpha}) \right) \\ = \bar{\psi}^0 \gamma^\alpha \psi^0 + \frac{i}{2} C_\lambda^{\rho\sigma} x^\lambda \overline{D_\sigma \psi^0} \gamma^\alpha (D_\rho \psi^0)^0 + ia (\bar{\psi}^0 \gamma^\alpha (D_0 \psi)^0 - \overline{D_0 \psi^0} \gamma^\alpha \psi^0) + \\ + \frac{ia}{2} \delta_0^\alpha (\bar{\psi}^0 \gamma^0 (D_0 \psi)^0 - \overline{D_0 \psi^0} \gamma^0 \psi^0). \end{aligned} \quad (4.87)$$

Using either the equations of motions for fermionic fields (4.86) or the expanded action (4.85) we can calculate the conserved  $U(1)$  current up to first order in  $a$ . Up to total derivative terms, we obtain:

$$\begin{aligned} j^0 &= \bar{\psi}^0 \gamma^0 \psi^0 - \frac{a}{2} x^j F_{j\sigma}^0 \bar{\psi}^0 \gamma^\sigma \psi^0 - \frac{ia}{2} \bar{\psi}^0 \gamma^j D_j^0 \psi^0, \\ j^k &= \bar{\psi}^0 \gamma^k \psi^0 + \frac{a}{2} x^k F_{0\sigma}^0 \bar{\psi}^0 \gamma^\sigma \psi^0 + \frac{ia}{2} \overline{D_0 \psi^0} \gamma^k \psi^0. \end{aligned} \quad (4.88)$$

Following the approach of [31] we discuss deformed dispersion relations for our fields. Looking at (4.84) we conclude that there is no modification in the dispersion

relation for the photon field  $A_\mu^0$ . On the other hand, collecting only the terms quadratic in the fermionic field from (4.85) we obtain the following equation of motion

$$i\gamma^\mu \partial_\mu \psi + a\gamma^j \partial_0 \partial_j \psi = 0. \quad (4.89)$$

Assuming a plane wave solution and inserting it in (4.89) leads to

$$(k^0)^2 - (1 - ak_0)^2 \vec{k}^2 = 0, \quad (4.90)$$

or expanded up to first order in the deformation parameter  $a$

$$(k^0)^2 - \vec{k}^2 + 2ak^0 \vec{k}^2 = 0. \quad (4.91)$$

The dispersion relation is modified in the same way for all the directions of motion and there is no birefringence effect. Assuming that  $\vec{k} = k\vec{e}_z$  we obtain the following group velocity

$$v_g = \frac{\partial k_0}{\partial k} = 1 - 2ak. \quad (4.92)$$

The group velocity<sup>8</sup> for the massless fermions has momentum dependence in accordance with the previously obtained results [32]. Note that this is the first time (to the best of our knowledge) that the dispersion follows from the (effective) action and the corresponding equations of motion.

However, the phenomenological consequences of our model should be taken with care. It is important to remember that the corresponding symmetry of the twisted  $\kappa$ -Minkowski space is the twisted  $igl(1,3)$  symmetry, and not the  $\kappa$ -Poincaré symmetry. In our previous analysis [14] we kept the  $\kappa$ -Poincaré symmetry and that resulted in no deformation of the dispersion relations in first order in the deformation parameter, see analysis in [34]. Also, the  $x$ -dependent terms in our expanded action clearly demand better understanding, possibly in terms of geometric degrees of freedom. As in any field theory, one needs to understand the renormalization properties of the theory before making any predictions. Based on the results obtained in field theory on the canonically deformed space-time one does expect additional terms in the action which render theory renormalizable [22, 31]. Finally, the second order corrections in deformation parameter might turn out to be essential for deforming of the dispersion relations.

## 5 Conclusion and outlook

In this paper we used the twist formalism to gain a better understanding of the gauge theory on  $\kappa$ -Minkowski space-time and to resolve certain ambiguities we encounter in

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<sup>8</sup>It is possible to make judicious choice of the deformation vector  $a^\mu$  and thus change the sign on the rhs of (4.92). This choice gives the group velocity bigger than the speed of light (we used natural units, so  $c = 1$ ). In view of the latest experimental results from OPERA collaboration [33] this result could be interesting. However, since neutrinos do not couple to the electromagnetic field, one would have to construct a non-Abelian gauge theory model and check what happens with the dispersion relations in that case.

our previous analysis [14]. The twist formalism provided us with a naturally defined differential calculus. As a consequence, we obtained uniquely defined derivatives, thus solving one ambiguity. Next, in the twist approach the integral has the trace property, and there is no need to introduce an additional measure function in the integral. This also means that the limit of vanishing deformation parameter  $a$  reproduces the undeformed case without the need for additional field redefinitions. One puzzling feature of the gauge field on  $\kappa$ -Minkowski disclosed within the formalism introduced in [16], was that a gauge field is given in terms of the higher order differential operator. This produced "torsion-like" terms in the field strength which were simply omitted in the constructed action. In the twist approach however, the commutation rule of basis 1-forms with functions reproduces the effect of "higher order differential operator" gauge field without producing unwanted terms in the action. Furthermore, the gauge field is enveloping algebra-valued, and one needs to use the Seiberg-Witten map to express noncommutative variables (gauge parameter, fields) in terms of the commutative ones thus keeping the same number of degrees of freedom as in the commutative case (where the degrees of freedom are Lie algebra-valued). This mapping introduces additional ambiguities in the construction of the effective model. In the previous analysis we used the additional symmetry requirements to fix these ambiguities in the constructed action, while in the twist approach the ambiguities were completely absent. In a way, this might have been expected since in the twist approach one deforms the symmetry of the theory first, and then consistently applies the consequences of this deformation on the space-time itself.

We have shown that the twisting of symmetries, as a way of deforming the algebra of coordinates, is compatible with the local gauge principle. From this point of view, one could interpret our (expanded) model as an example of non-local and/or non-linear extensions of electrodynamics obtained from a more fundamental theory. The obstruction we have encountered in the construction of Hodge-dual field-strength tensor is a manifestation of the fact that the introduction of a noncommutative geometrical structure prevents decoupling of translation and gauge symmetries, similarly as in general relativity where translations are part of "gauge" symmetries (diffeomorphism group). Namely, if one sees electric charge and magnetic flux as fundamental quantities in electrodynamics, the Maxwell equations can be written in the metric-free form. Only the relation between the flows of electric charge and magnetic flux, which includes the Hodge dual field strength, introduces the metric degrees of freedom in the local gauge theory [35]. Although the mixing of space-time and internal symmetries appeared as a problem in our construction, this is in fact one of the most intriguing property of models based on non-trivial algebras of coordinates. One possible way to understand this mixing was offered in the framework of Yang-Mills type matrix models [36]. There it was shown that  $U(1)$  part of general  $U(N)$  gauge group can be interpreted as induced gravity coupling to the rest ( $SU(N)$ ) gauge degrees of freedom. It would be interesting to see if such an interpretation is possible in our framework, by constructing models with larger gauge groups.

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## A Manipulations with spinors

Trace over spinor indices I:

$$\begin{aligned}
S_m &= \int \left( (\overline{D\psi})_B \star \psi_A - \bar{\psi}_B \star (D\psi)_A \right) \wedge_\star (V \wedge_\star V \wedge_\star V \gamma_5)_{BA} \\
&= \int \left( -(\bar{R}^\alpha \psi)_A \star (\bar{R}_\alpha \overline{D\psi})_B + (\bar{R}^\alpha D\psi)_A \star (\bar{R}_\alpha \bar{\psi})_B \right) \wedge_\star (V \wedge_\star V \wedge_\star V \gamma_5)_{BA} \\
&= \text{Tr} \int \left( (\bar{R}^\alpha D\psi) \star (\bar{R}_\alpha \bar{\psi}) - (\bar{R}^\alpha \psi) \star (\bar{R}_\alpha \overline{D\psi}) \right) \wedge_\star (V \wedge_\star V \wedge_\star V \gamma_5)
\end{aligned}$$

The minus sign comes from commuting the spinor fields.

Trace over spinor indices II:

$$\begin{aligned}
S_{m2} &= -c_2 \int \bar{\psi}_B \star (D\psi)_A \wedge_\star (V \wedge_\star V \wedge_\star V \gamma_5)_{BA} \\
&= -c_2 \int \bar{\psi}_B \star (D_{\mu_1}^\star \psi)_A \star dx^{\mu_1} \wedge_\star (\gamma_{\mu_2} dx^{\mu_2} \wedge_\star \gamma_{\mu_3} dx^{\mu_3} \wedge_\star \gamma_{\mu_4} dx^{\mu_4} \gamma_5)_{BA} \\
&= -c_2 \int \bar{\psi}_B \star (D_{\mu_1}^\star \psi)_A \star (\gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} \gamma_5)_{BA} \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} d^4 x \\
&= -c_2 \int \bar{\psi}_B \star (D_{\mu_1}^\star \psi)_A \star \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} i \epsilon_{\mu_2 \mu_3 \mu_4 \mu_5} (\gamma^{\mu_5})_{BA} d^4 x \\
&= -6c_2 i \int \bar{\psi}_B \gamma_{BA}^{\mu_1} \star (D_{\mu_1}^\star \psi)_A \star d^4 x \\
&= -6c_2 i \int \bar{\psi} \gamma^\mu \star (D_\mu^\star \psi) \star d^4 x.
\end{aligned}$$

In the second line we used:

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} = \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} d^4 x, \quad \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\mu_5 \mu_2 \mu_3 \mu_4} = -6\delta_{\mu_5}^{\mu_1},$$

and in the third line we used identity valid in four dimensions:

$$\gamma_\mu \gamma_\nu \gamma_\rho = (\eta_{\mu\nu} \eta_{\rho\sigma} - \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) \gamma^\sigma + i \epsilon_{\mu\nu\rho\sigma} \gamma^\sigma \gamma_5.$$

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